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Conformal transformation of Douglas space of second kind with special (α, β)-metric

Rishabh Ranjan
Department of Mathematics and Statistics, Sam Higginbottom University of Agriculture, Technology and Sciences, Prayagraj, India
P.N. Pandey
Department of Mathematics, University of Allahabad, Prayagraj, India, and
Ajit Paul
Department of Mathematics and Statistics, Sam Higginbottom University of Agriculture, Technology and Sciences, Prayagraj, India

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Abstract

Purpose – In this paper, the authors prove that the Douglas space of second kind with a generalised form of special (α, β)-metric F, is conformally invariant.

Design/methodology/approach – For, the authors have used the notion of conformal transformation and Douglas space.

Findings – The authors found some results to show that the Douglas space of second kind with certain (α, β)-metrics such as Randers metric, first approximate Matsumoto metric along with some special (α, β)-metrics, is invariant under a conformal change.

Originality/value – The authors introduced Douglas space of second kind and established conditions under which it can be transformed to a Douglas space of second kind.

Keywords Randers metric, Douglas space, Berwald space, Conformally invariant

Paper type Research paper

1. Introduction

A number of geometers have been studying Douglas space [1, 2] from different point of view. The theory of Finsler spaces more precisely Berwald spaces with an (α, β)-metric [3–5] have significant role to develop the Finsler geometry [6]. The concept of Douglas space of second kind with (α, β)-metric was first discussed by I. Y. Lee [7] in Finsler geometry. In [8], S. Bacso and Matsumoto developed the concept of Douglas space as an extension of Berwald space. In [9], S. Bacso and Szilagyi introduced the concept of weakly-Berwald space as another extension of Berwald space. In [10], M. S. Kneblman started working on the concept of conformal Finsler spaces and consequently, this notion was explored by M. Hashiguchi [11]. In [12, 13] Y. D. Lee and B.N. Prasad developed the conformally invariant tensorial quantities in a Finsler space with (α, β)-metric under conformal β-change.

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In this paper, we prove that the Douglas space of second kind with generalised special (α, β)-metric is conformally invariant. In the consequence, we find some results to show that the Douglas space of second kind with certain (α, β)-metric such as Randers metric, first approximate Matsumoto metric and Finsler space with some generalised form of (α, β)-metric remains unchanged geometrically under a confomal transformation.

2. Preliminaries

A Finsler space $F^n = (M, F(\alpha, \beta))$ is said to be with an (α, β)-metric if $F(\alpha, \beta)$ is a positively homogeneous function in α and β of degree 1, where α is Riemannian metric given by $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$ is 1-form. The space $R^n = (M, \alpha)$ is called Riemannian space associated with F^n. We shall use the following symbols [6];

$$b^i = a^{ir} b_r,\ \ b^2 = a^{rs} b_r b_s$$

$$2r_{ij} = b_{i|j} + b_{j|i},\ \ 2s_{ij} = b_{i|j} - b_{j|i}$$

$$s^i_j = a^{ir} s_{rj},\ \ s_j = b_r s^r_j$$

The Berwald connection

$$B\Gamma = \left\{ G^i_{jk}(x, y), G^i_j \right\}$$

of F^n plays an important role in this paper. B^i_{jk} denotes the difference tensor of G^i_{jk} and γ^i_{jk} that is

$$G^i_{jk}(x, y) = \gamma^i_{jk}(x) + B^i_{jk}(x, y). \tag{1}$$

Using the subscript 0 and transvecting by y^i, we get

$$G^i_j = \gamma^i_{0j} + B^i_j \ \ and \ \ 2G^i = \gamma^i_{00} + 2B^i, \tag{2}$$

and then $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$. A Finsler space F^n of dimension n is called a Douglas space [14] if

$$D^{ij} = G^i(x, y)Y^i - G^j(x, y)y^i, \tag{3}$$

are homogeneous polynomial of (y^i) of degree three.

Next, differentiating (3) with respect to y^m, we obtain the following definitions;

Definition 1. *([14]) A Finsler space F^n is a Douglas space of second kind if $D^i_{im} = (n+1)G^i - G^{im}_m y^i$ is a two homogeneous polynomial in (y^i).*

On the other hand, a Finsler space with (α, β)-metric is a Douglas space of second kind if and only if

$$B^{im}_m = (n+1)B^i - B^m_m y^i, \tag{4}$$

are homogeneous equation in (y^i) of degree two, when B^m_m is same as given in [14].

Furthermore, differentiating Eqn (4) with respect to y^h, y^j *and* y^k, we obtain

$$B^{im}_{hjkm} = B^i_{hjk} = 0. \tag{5}$$

Definition 2. *A Finsler space F^n with (α, β)-metric is known as Douglas space of second kind if $B^{im}_m = (n+1)B^i - B^m_m y^i$ is a homogeneous polynomial in (y^i) of degree two.*

3. Douglas space of second kind with (α, β)-metric

Under this section, we discuss the criteria for a Finsler space with an (α, β)-metric to be a Douglas space of second kind [2].

The spray coefficient $G^i(x, y)$ of F^n can be expressed as [4].

$$2G^i = \gamma^i_{00} + 2B^i \tag{6}$$

$$B^i = \frac{\alpha F_\beta}{F_\alpha} s^i_0 + C^* \left[\frac{\beta F_\beta}{\alpha F} y^i - \frac{\alpha F_{\alpha\alpha}}{F_\alpha} \left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right], \tag{7}$$

where

$$C^* = \frac{\alpha\beta\left(r_{00}F_\alpha - 2\alpha s_0 F_\beta\right)}{2\left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right)},$$

$$\gamma^2 = b^2\alpha^2 - \beta^2. \tag{8}$$

Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ is hp(2), Eqn (7) yields

$$B^{ij} = \frac{\alpha F_\beta}{F_\alpha}\left(s^i_0 y^j - s^j_0 y^i\right) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^* \left(b^i y^j - b^j y^i\right). \tag{9}$$

By means of (3) and (9), we obtain the following lemma [14];

Lemma 1. *A Finsler space F^n with an (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are hp(3).*

Differentiating (9) with respect to y^h, y^k, y^p and y^q, we can have $D^{ij}_{hkpq} = 0$ which are equivalent to $D^{im}_{hkpm} = (n+1)D^i_{hkp} = 0$. Hence, a Finsler space F^n satisfying the condition $D^{ij}_{hkpq} = 0$ is called Douglas space. Now, differentiating Eqn (9) with respect to y^m and contracting m and j in the resulting equation, we get

$$\begin{aligned} B^{im} = {} & \frac{(n+1)\alpha F_\beta s^i_0}{F_\alpha} + \frac{\alpha\left\{(n+1)\alpha^2 \Omega F_{\alpha\alpha} b^i + \beta\gamma^2 A y^i\right\} r_{00}}{2\Omega^2} \\ & - \frac{\alpha^2\left\{(n+1)\alpha^2 \Omega F_\beta F_{\alpha\alpha} b^i + B y^i\right\} s_0}{F_\alpha \Omega^2} - \frac{\alpha^3 F_{\alpha\alpha} y^i r_0}{\Omega} \end{aligned} \tag{10}$$

where $\Omega = \left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right)$, provided that $\Omega \neq 0$, $A = \alpha F_\alpha F_{\alpha\alpha\alpha} + 3F_\alpha F_{\alpha\alpha} - 3\alpha (F_{\alpha\alpha})^2$ and

$$B = \alpha\beta\gamma^2 F_\alpha F_\beta F_{\alpha\alpha\alpha} + \beta\left\{\left(3\gamma^2 - \beta^2\right)F_\alpha - 4\alpha\gamma^2 F_{\alpha\alpha}\right\} F_\beta F_{\alpha\alpha} + \Omega F F_{\alpha\alpha} \tag{11}$$

Following result is used in the succeeding section [7]:

Theorem 1. *A Finsler space F^n is a Douglas space if second kind if and only if B^{im}_m are homogeneous polynomials in (y^m) of degree two, where B^{im}_m is given by Eqs (10) and (11), provided $\Omega \neq 0$.*

4. Conformal change of Douglas space of second kind with (α, β)-metric

In this section, we find the criteria for a Douglas space of second kind to be conformally invariant.

Let $F^n = (M, F)$ and $\overline{F}^n = (M, \overline{F})$ be two Finsler spaces. Then F^n is called conformal to $\overline{F}^n$ if we have a function $\sigma(x)$ in each coordinate neighbourhood of M^n such that $\overline{F}(x, y) = e^\sigma F(x, y)$ and this transformation $F \to \overline{F}$ is called conformal change.

A conformal change of (α, β)-metric is given as $(\alpha,\beta) \to (\overline{\alpha},\overline{\beta})$, where $\overline{\alpha} = e^{\sigma}\alpha$, $\overline{\beta} = e^{\sigma}\beta$ that is,

$$\overline{a}_{ij} = e^{2\sigma}a_{ij},\ \overline{b}_i = e^{\sigma}b_i \tag{12}$$

$$\overline{a}^{ij} = e^{-2\sigma}a^{ij},\ \overline{b}^{i} = e^{-\sigma}b^{i} \tag{13}$$

and $b^2 = a^{ij}b_ib_j = \overline{a}^{ij}\overline{b}_i\overline{b}_j$

From Eqn (13), the Christoffel symbols are given by:

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}, \tag{14}$$

Where, $\sigma_j = \partial_j\sigma$ and $\sigma^i = a^{ij}\sigma_j$.

Using (13) and (14), we obtain the following identities:

$$\overline{\nabla}_j\overline{b}_i = e^{\sigma}(\nabla_j b_i + \rho a_{ij} - \sigma_i b_j),$$

$$\overline{r}_{ij} = e^{\sigma}\left[r_{ij} + \rho a_{ij} - \frac{1}{2}(b_i\sigma_j + b_j\sigma_i)\right],$$

$$\overline{s}_{ij} = e^{\sigma}\left[s_{ij} + \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)\right],$$

$$\overline{s}^i_j = e^{-\sigma}\left[s^i_j + \frac{1}{2}(b^i\sigma_j - b_j\sigma^i)\right],$$

$$\overline{s}_j = s_j + \frac{1}{2}(b^2\sigma_j - \rho b_j), \tag{15}$$

Where, $\rho = \sigma_r b^r$.

Using Eqs (14) and (15), we get easily the followings:

$$\overline{\gamma}^i_{00} = \gamma^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma_j, \tag{16}$$

$$\overline{r}_{00} = e^{\sigma}(r_{00} + \rho\alpha^2 - \sigma_0\beta), \tag{17}$$

$$\overline{s}^i_0 = e^{-\sigma}\left[s^i_0 + \frac{1}{2}(\sigma s_0 b^i - \beta\sigma^i)\right], \tag{18}$$

$$\overline{s}_0 = s_0 + \frac{1}{2}(\sigma_0 b^i - \rho\beta). \tag{19}$$

Now we obtain the conformal transformation of B^{ij} given by Eqn (9).

Consider $\overline{F}(\alpha,\beta) = e^{\sigma}F(\alpha,\beta)$ then

$$\overline{F}_{\overline{\alpha}} = F_{\alpha},\ \overline{F}_{\overline{\alpha}\overline{\alpha}} = e^{-\sigma}F_{\alpha\alpha},\ \overline{F}_{\overline{\beta}} = F_{\beta},\ \overline{\gamma}^2 = e^{2\alpha}\gamma^2 \tag{20}$$

From Eqs (8), (19), (20) and using Theorem 3.1, we obtain

$$\overline{C}^{*} = e^{\sigma}(C^{*} + D^{*}), \tag{21}$$

Where,

$$D^{*} = \frac{\alpha\beta\left[(\beta\alpha^2 - \sigma_0\beta)F_{\alpha} - \alpha(b^2\sigma_0 - \rho\beta)F_{\beta}\right]}{2(\beta^2F_{\alpha} + \alpha\gamma^2F_{\alpha\alpha})} \tag{22}$$

Hence B^{ij} can be expressed as:

$$\overline{B}^{ij} = \frac{\alpha F_\beta}{F_\alpha}\left(s_0^i y^j - s_0^j y^i\right) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^*\left(b^i y^j - b^j y^i\right)$$

$$+\left(\frac{\alpha\sigma_0 F_\beta}{F_\alpha} + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} D^*\right)\left(b^i y^j - b^j y^i\right) - \frac{\alpha\beta F_\beta}{2F_\alpha}\left(\sigma^i y^j - \sigma^j y^i\right),$$

$$= B^{ij} + C^{ij},$$

Where,

$$C^{ij} = \left(\frac{\alpha\sigma_0 F_\beta}{F_\alpha} + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} D^*\right)\left(b^i y^j - b^j y^i\right) - \frac{\alpha\beta F_\beta}{2F_\alpha}\left(\sigma^i y^j - \sigma^j y^i\right).$$

Using Eqn (11), we can have

$$\overline{\Omega} = e^{2\alpha}\Omega, \;\; \overline{A} = e^{-\sigma} A, \;\; \overline{B} = e^{2\alpha} B. \tag{23}$$

Now, we use conformal transformation on B^{im}_m and obtain

$$\overline{B}^{im}_m = B^{im}_m + K^{im}_m \tag{24}$$

Where, K^{im}_m is given by [15, 16].

$$\begin{aligned} 2K^{im}_m = \; & \frac{(n+1)\alpha F_\beta}{F_\alpha}\left(\sigma_0 b^i - \beta\sigma^i\right) + \alpha\left\{\frac{(n+1)\alpha^2\Omega F_{\alpha\alpha} b^i + \beta\gamma^2 A y^i}{\Omega^2}\right\}\left(\rho\alpha^2 - \sigma_0\beta\right) \\ & - \left[\frac{\alpha^2\{(n+1)\alpha^2\Omega\} F_\beta F_{\alpha\alpha} b^i + B y^i}{F_\alpha \Omega^2}\right]\left(b^2\sigma_0 - \rho\beta\right). \end{aligned} \tag{25}$$

Therefore, we obtain the following result:

Theorem 2. *A Douglas space of second kind is conformally invariant if and only if $K^{im}_m(x)$ are homogeneous polynomial in (y^i) of degree two.*

5. Conformal change of Douglas space of second kind with special (α, β)-metric $F = \alpha + \epsilon\beta + k\frac{\beta^{t+1}}{\alpha^t}$

Consider a Finsler manifold with special (α, β)-metric defined as

$$F = \alpha + \epsilon\beta + k\frac{\beta^{t+1}}{\alpha^t},$$

Where, ϵ and k are constant.

Then we obtain

$$\begin{aligned} F_\alpha &= 1 - tk\frac{\beta^{t+1}}{\alpha^{t+1}}, \\ F_\beta &= \epsilon + k(t+1)\frac{\beta^t}{\alpha^t}, \\ F_{\alpha\alpha} &= t(t+1)k\frac{\beta^{t+1}}{\alpha^{t+2}} \\ F_{\alpha\alpha\alpha} &= \frac{-6k\beta^2}{\alpha^4}. \end{aligned} \tag{26}$$

Therefore, using Eqn (11), we obtain

$$\Omega = \frac{-t(t+2)k\beta^{t+3} + \left[\alpha^t\beta + b^2t(t+1)\alpha\beta^t\right]\alpha\beta}{\alpha^{t+1}}$$
$$A = t(t+1)k\frac{\beta^{t+1}}{\alpha^{t+2}}\left[(1-t) - 2t(t+2)k\frac{\beta^{t+1}}{\alpha^{t+1}}\right] \tag{27}$$
$$B = \prod_1 + \prod_2 + \prod_3$$

Where,

$$\prod_1 = -t(t+1)(t+2)k\frac{\beta^{t+2}}{\alpha^{t+2}}\left[\epsilon + k(t+1)\frac{\beta^t}{\alpha^t} - \epsilon nk\frac{\beta^{t+1}}{\alpha^{t+1}} - t(t+1)k^2\frac{\beta^{2t+1}}{\alpha^{2t+1}}\right]\left(b^2\alpha^2-\beta^2\right),$$

$$\prod_2 = t(t+1)k\frac{\beta^{t+2}}{\alpha^{t+2}}\left(\epsilon + k(t+1)\frac{\beta^t}{\alpha^t}\right)\left[\left(3-t(4t+7)\frac{\beta^{t+1}}{\alpha^{t+1}}\right)b^2\alpha^2 + \left(t(t+2)k\frac{\beta^{t+1}}{\alpha^{t+1}} - 1\right)4\beta^2\right],$$

$$\begin{aligned}\prod_3 = {} & t(t+1)k\frac{\beta^{t+2}}{\alpha^{t+2}}\Bigg[\left(\alpha\beta+\varepsilon\beta^2\right) + t(t+1)\frac{\beta^t}{\alpha^t}\Big\{\left(b^2\alpha^2+\varepsilon b^2\alpha\beta - k\beta^2 - \varepsilon k\beta^3\alpha^{-1}\right) \\ & + \frac{\beta^{t+1}}{\alpha^{t+1}}\left(b^2\alpha^2 - k^2\beta^2\right)\Big\}\Bigg],\end{aligned}$$

Hence, using Eqn (26), K_m^{im} can be reduced as

$$\begin{aligned}2K_m^{im} = {} & (n+1)\alpha\left[\epsilon + k(t+1)\frac{\beta^t}{\alpha^t}\right]\left(\sigma_0 b^i - \beta\sigma^i\right) + (\alpha A_1 + \alpha A_2)\left(\rho\alpha^2 - \sigma_0\beta\right) \\ & -\left[B_0 + (B_1+B_2+B_3)y^i - C_1\right]\left(b^2\sigma_0 - \rho\beta\right).\end{aligned} \tag{28}$$

Where,

$$\alpha A_1 = \frac{(n+1)t(t+1)k\alpha^2\beta^{t+1}}{\left\{\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1}\right\} - t(t+2)k\beta^{t+3}}b^i,$$

$$\alpha A_2 = \frac{t(t+1)k\left[(1-t)\alpha^{t+1} - 2t(t+2)k\beta^{t+2}\right]\beta^t\gamma^2}{\left[\left\{\alpha^{t+1}\beta + b^2t(t+1)\alpha^2\beta^t\right\} - t(t+2)k\beta^{t+2}\right]^2}y^i$$

$$B_0 = \frac{(n+1)t(t+1)k\alpha^4\beta^t\left(\varepsilon\alpha^t + k(t+1)\beta^t\right)}{\left(\alpha^{t+1} - tk\beta^{t+1}\right)\left[\alpha^{t+1}\beta + b^2t(t+1)\alpha^2\beta^t - t(t+2)k\beta^{t+2}\right]}b^i$$

$$B_1 = \frac{-t(t+1)(t+2)k\alpha^2\beta^{t+2}\left(\epsilon\alpha^{2t+1} + k(t+1)\alpha^{t+1}\beta^t - \varepsilon tk\alpha^t\beta^{t+1} - t(t+1)k^2\beta^{2t+1}\right)}{\left(\alpha^{t+1} - tk\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}\right]^2}\gamma^2,$$

$$B_2 = \frac{t(t+1)(t+2)k\alpha^2\beta^{t+2}\left(\varepsilon\alpha^t + k(t+1)\beta^t\right)}{\left(\alpha^{t+1} - tk\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}\right]^2}$$
$$\left[3b^2\alpha^{t+3} - t(4t+7)kb^2\alpha^2\beta^{t+1} - 4\alpha^{t+1}\beta^2 + 4t(t+2)k\beta^{t+3}\right].$$

$$B_3 = \frac{kt(t+1)(\alpha\beta)^{t+2}}{\left(\alpha^{t+1} - tk\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}\right]^2}$$
$$\left[\alpha^{t+2}\beta + \epsilon\alpha^{t+1}\beta^2 t(t+1)\left(b^2\alpha^3\beta^t - k\alpha\beta^{t+2}\right) + \left(t(t+1) + \epsilon b^2\right)\alpha^2\beta^{t+2}\right.$$
$$\left. - \left(t(t+1)k\epsilon + k^2\right)\beta^{t+3}\right]$$

$$C = \frac{-t(t+1)k\alpha^2\beta^{t+1}}{\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}\right]}y^i.$$

Now, Eqn (28) can also be written as

$$2K^{im}_m = (n+1)\alpha\left[\varepsilon + k(t+1)\left(\alpha^{-1}\beta\right)\right]\left(\sigma_0 b^i - \beta\sigma^i\right) + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7. \quad (29)$$

where,

$$p_1 = \alpha A_1\left(\rho\alpha^2 - \sigma_0\beta\right)$$
$$p_2 = \alpha A_2\left(\rho\alpha^2 - \sigma_0\beta\right)$$
$$p_3 = -B_0\left(b^2\sigma_0 - \rho\beta\right)$$
$$p_4 = -B_1 y^i\left(b^2\sigma_0 - \rho\beta\right)$$
$$p_5 = -B_2 y^i\left(b^2\sigma_0 - \rho\beta\right)$$
$$p_6 = -B_3 y^i\left(b^2\sigma_0 - \rho\beta\right)$$
$$p_7 = C\left(b^2\sigma_0 - \rho\beta\right)$$

showing that K^{im}_m is homogeneous polynomial of degree 2 in y^i.

Theorem 3. *A Douglas space of second kind with special* (α, β)*-metric* $F = \alpha + \epsilon\beta + k\frac{\beta^{t+1}}{\alpha^t}$, *where* ϵ *and* k *are constants, is conformally invariant.*

With the help of Theorem 3 it can be proved that a Douglas space of second kind with a Finsler space of certain (α, β)-metric is conformally transformed to a Douglas space of second kind. In this way, one can have following possible cases;

Case(i). If $\epsilon = 1$ and $k = 0$, we have $F = \alpha + \beta$ which is Randers metric. In case, $2K^{im}_m$ occupies the form

$$2K^{im}_m = (n+1)\alpha\left(\sigma_0 b^i - \beta\sigma^i\right), \quad (30)$$

Which shows K^{im}_m is homogeneous polynomial in (y^i) of degree two.

Note that in this case, $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 0$.

Corollary 1. *A Douglas space of second kind with Randers metric $F = \alpha + \beta$, is conformally invariant.*

Case(ii). If $\epsilon = 0$ and $k = 1$, we have $F = \alpha + \frac{\beta^{t+1}}{\alpha^t}$. In this case $2K^{im}_m$ obtains the form

$$2K^{im}_m = (n+1)(t+1)(\alpha^{-1}\beta)\alpha\left(\sigma_0 b^i - \beta\sigma^i\right) + q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7, \quad (31)$$

Where,

$$q_1 = \frac{(n+1)t(t+1)\alpha^2\beta^{t+1}}{\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}} b^i\left(\sigma_0 b^i - \beta\sigma^i\right),$$

$$q_2 = \frac{t(t+1)\left[(1-t)\alpha^{t+1} - 2t(t+2)\beta^{t+1}\right]\beta^t\gamma^2}{\left[\alpha^{t+1}\beta + b^2t(t+1)\alpha^2\beta^t - t(t+2)\beta^{t+2}\right]^2}\left(\rho\alpha^2 - \sigma_0\beta\right),$$

$$q_3 = \frac{(n+1)t(t+1)^2\alpha^4\beta^{2t}b^i}{\left(\alpha^{t+1} - t\beta^{t+1}\right)\left[\alpha^{t+1}\beta + b^2t(t+1)\alpha^2\beta^t - t(t+2)\beta^{t+2}\right]}\left(b^2\sigma_0 - \rho\beta\right),$$

$$q_4 = \frac{t(t+1)^2(t+2)\alpha^2\beta^{2t+2}\gamma^2}{\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)\beta^{t+3}\right]^2}\left(b^2\sigma_0 - \rho\beta\right),$$

$$q_5 = \frac{-t(t+1)^2\alpha^2\beta^{2t+2}\left[3b^2\alpha^{t+3} - t(4t+7)b^2\alpha^2\beta^{t+1} - 4\alpha^2\beta^{t+1} + 4t(t+2)\beta^{t+3}\right]}{\left(\alpha^{t+1} - t\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}\right]^2} y^i\left(b^2\sigma_0 - \rho\beta\right),$$

$$q_6 = \frac{-t(t+1)(\alpha\beta)^{t+2}\left[\alpha^{t+2}\beta + t(t+1)\left\{b^2\alpha^3\beta^t + \alpha^2\beta^{t+1} - \alpha\beta^{t+2}\right\} - \beta^{t+3}\right]}{\left(\alpha^{t+1} - t\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}\right]^2} y^i\left(b^2\sigma_0 - \rho\beta\right),$$

$$q_7 = \frac{t(t+1)\alpha^2\beta^{t+1}}{\alpha^{t+1}\beta^2 + b^2t(t+1)\alpha^2\beta^{t+1} - t(t+2)k\beta^{t+3}} y^i\left(b^2\sigma_0 - \rho\beta\right),$$

Showing that K^{im}_m is homogeneous polynomial in (y^i) of degree 2.
Thus, we can have following;

Corollary 2. *A Douglas space of second kind with special (α, β)-metric $F = \alpha + \frac{\beta^{t+1}}{\alpha^t}$ is conformally transformed to a Douglas space of second kind.*

Case(iii). If $\epsilon = 1$ and $k = 1$, we obtain $F = \alpha + \beta + \frac{\beta^{t+1}}{\alpha^t}$. In the case, $2K^{im}_m$ occupies the form

$$2K^{im}_m = (n+1)\left[1 + (t+1)\left(\alpha^{-1}\beta\right)\right]\alpha\left(\sigma_0 b^i - \beta\sigma^i\right) + r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7, \quad (32)$$

where,

$$r_1 = \frac{(n+1)t(t+1)\alpha^2\beta^{t+1}}{\alpha^{t+1}\beta + b^2t(t+1)\alpha^2\beta^t - t(t+2)\beta^{t+2}} b^i\left(\rho\alpha^2 - \sigma_0\beta\right),$$

$$r_2 = \frac{t(t+1)\left[(1-t)\alpha^{t+1} - 2t(t+2)\beta^{t+1}\right]\beta^t\gamma^2}{\left[\alpha^{t+1}\beta + b^2t(t+1)\alpha^2\beta^t - t(t+2)\beta^{t+2}\right]^2} y^i\left(\rho\alpha^2 - \sigma_0\beta\right),$$

$$r_3 = \frac{-(n+1)t(t+1)\alpha^4\beta^t\left(\alpha^t+(t+1)\beta^t\right)b^i}{\left(\alpha^{t+1}-t\beta^{t+1}\right)\left[\alpha^{t+1}\beta+b^2t(t+1)\alpha^2\beta^t-t(t+2)\beta^{t+2}\right]}\left(b^2\sigma_0-\rho\beta\right),$$

$$r_4 = \frac{t(t+1)(t+2)\alpha^2\beta^{t+2}\left(\alpha^{2t+1}+(t+1)\alpha^{t+1}\beta^t-t\alpha^t\beta^{t+1}-t(t+1)\beta^{2t+1}\right)\gamma^2}{\left(\alpha^{t+1}-t\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2+b^2t(t+1)\alpha^2\beta^{t+1}-t(t+2)\beta^{t+3}\right]^2}y^i\left(b^2\sigma_0-\rho\beta\right),$$

$$r_5 = \frac{-t(t+1)\alpha^2\beta^{t+2}\left(\alpha^t+(t+1)\beta^t\right)}{\left(\alpha^{t+1}-t\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2+b^2t(t+1)\alpha^2\beta^{t+1}-t(t+2)\beta^{t+3}\right]^2}\left[3b^2\alpha^{t+3}-t(4t+7)b^2\alpha^2\beta^{t+1}\right.$$
$$\left.-4\alpha^{t+1}\beta^2+4t(t+2)\beta^{t+3}\right]y^i\left(b^2\sigma_0-\rho\beta\right),$$

$$r_6 = \frac{t(t+1)(\alpha\beta)^{t+2}}{\left(\alpha^{t+1}-t\beta^{t+1}\right)\left[\alpha^{t+1}\beta^2+b^2t(t+1)\alpha^2\beta^{t+1}-t(t+2)\beta^{t+3}\right]^2}\left[\alpha^{t+2}\beta+\alpha^{t+1}\beta^2\right.$$
$$\left.+t(t+1)\left(b^2\alpha^3\beta^t-\alpha\beta^{t+2}\right)+\left(t^2+1+b^2\right)\alpha^2\beta^{t+1}-\left(t^2+t+1\right)\beta^{t+3}\right]y^i\left(b^2\sigma_0-\rho\beta\right),$$

$$r_7 = \frac{-t(t+1)\alpha^2\beta^t}{\alpha^{t+1}\beta+b^2t(t+1)\alpha^2\beta^t-t(t+2)\beta^{t+2}}y^i\left(b^2\sigma_0-\rho\beta\right).$$

Showing that K^{im}_m is a homogeneous polynomial in (y^i) of degree 2.
Thus, we obtain the following;

Corollary 3. *A Douglas space of second kind with special* (α, β)*-metric* $F = \alpha+\beta+\frac{\beta^{t+1}}{\alpha^t}$ *is conformally invariant.*

Case(iv). If $\epsilon = 1, k = 1$ and $t = 1$, we obtain $F = \alpha+\beta+\frac{\beta^2}{\alpha}$. Then, $2K^{im}_m$ reduces in the form

$$2K^{im}_m = (n+1)\left[1+2\left(\alpha^{-1}\beta\right)\right]\alpha\left(\sigma_0b^i-\beta\sigma^i\right)+u_1+u_2+u_3+u_4+u_5+u_6+u_7, \quad (33)$$

Where,

$$u_1 = \frac{2(n+1)\alpha^2\beta}{\left(1+2b^2\right)\alpha^2-3\beta^2}b^i\left(\rho\alpha^2-\sigma_0\beta\right),$$

$$u_2 = \frac{12\beta\gamma^2}{\left[\left(1+2b^2\right)\alpha^2-3\beta^2\right]^2}y^i\left(\rho\alpha^2-\sigma_0\beta\right),$$

$$u_3 = \frac{-2(n+1)\alpha^4(\alpha+2\beta)}{\left(\alpha^2-\beta^2\right)\left[\left(1+2b^2\right)\alpha^2-3\beta^2\right]}b^i\left(b^2\sigma_0-\rho\beta\right),$$

$$u_4 = \frac{6\alpha^2\left(\alpha^3+2\alpha^2\beta-\alpha\beta^2-2\beta^3\right)\gamma^2}{\beta\left(\alpha^2-\beta^2\right)\left[\left(1+2b^2\right)\alpha^2-3\beta^2\right]^2}y^i\left(b^2\sigma_0-\rho\beta\right),$$

$$u_5 = \frac{-2\alpha^2(\alpha+2\beta)}{\beta(\alpha^2-\beta^2)\left[(1+2b^2)\alpha^2-3\beta^2\right]^2}\left[3b^2\alpha^4-(11b^2+4)\alpha^2\beta^2\right.$$
$$\left.+12\beta^4\right]y^i(b^2\sigma_0-\rho\beta),$$

$$u_6 = \frac{2\alpha^3}{(\alpha^2-\beta^2)\left[(1+2b^2)\alpha^2-3\beta^2\right]^2}\left[(1+2b^2)\alpha^3+(3+b^2)\alpha^2\beta\right.$$
$$\left.-\alpha\beta^2-3\beta^3\right]y^i(b^2\sigma_0-\rho\beta),$$

$$u_7 = \frac{-2\alpha^2}{(1+2b^2)\alpha^2-3\beta^2}y^i(b^2\sigma_0-\rho\beta).$$

Showing that K^{im}_m is a homogeneous polynomial in (y^i) of degree 2.

Thus, we can have the following;

Corollary 4. *A Douglas space of second kind with first approximate Matsumoto metric $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ is invariant under conformal change.*

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Further reading

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Corresponding author
Rishabh Ranjan can be contacted at: ranjanrishabh196@gmail.com